3-manifolds, q-series, and topological strings

Sunghyuk Park

Caltech

Geometry and Physics Seminar, Harvard CMSA
June 30, 2020

$\hat{Z}$

Predicted from physics [Gukov-Putrov-Vafa '16, Gukov-Pei-Putrov-Vafa '17], $\hat{Z}$ is a 3d TQFT that is related to various branches of mathematics. In particular,

- it is closely related to complex Chern-Simons theory
- as well as various quantum invariants (semisimple or non-semisimple) such as WRT invariants and ADO invariants.
- It is supposed to have a categorification,
- and it often has interesting modularity (quantum modularity).
- It can be interpreted as Rozansky-Witten partition function with certain non-compact target,
- or as open topological string partition function.

It is okay if these words don't make much sense to you. The point is that $\hat{Z}$ is an interesting invariant!

In this talk

I will give a gentle introduction to $\hat{Z}$ and what's known about it, mostly from the mathematical point of view. Along the way, I will highlight some recent developments (some of which I was involved in).

Outline

Part 0. Basic quantum topology

- Chern-Simons theory
- Some quantum link invariants

Part 1. $\hat{Z}$ and F_K

- $\hat{Z}$ for negative definite plumbed 3-manifolds
- $\hat{Z}$ for knot complements: F_K
- R-matrix, and F_K for positive braid knots and more

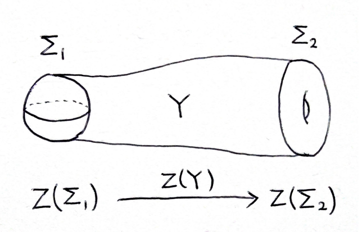
Part 2. Large N

- $\hat{Z}$ and F_K for $SU(N)$
- Large N limit of F_K and open topological strings

Part 0. Basic quantum topology

TQFT

A topological quantum field theory (TQFT) is a monoidal functor from the category of manifolds and their cobordisms to a monoidal category (typically the category of vector spaces).



When evaluated on a closed manifold, it evaluates to a number, and is called the partition function of the theory on the manifold.

Chern-Simons theory

Chern-Simons theory [Witten '89] is a 3d TQFT determined by a choice of gauge group G (compact semisimple Lie group) and an integer $k \in \mathbb{Z}_{>0}$ "level". Let's say $G = SU(2)$. For G-connections A on Y,

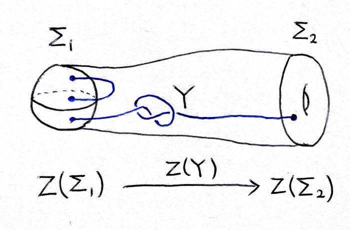
$$S(A) = \frac{k}{8\pi^2} \int_Y \operatorname{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right),$$

$$Z_{CS}(Y; G, k) \text{``}=\text{''} \int_{\mathcal{A}/\mathcal{G}} e^{2\pi i S(A)}\, dA.$$

Mathematical definition was given by [Reshetikhin-Turaev '91], using quantum groups. This Chern-Simons partition function Z_{CS} is a 3-manifold invariant called the Witten-Reshetikhin-Turaev (WRT) invariant.

Line operators in Chern-Simons theory

In Chern-Simons theory, we can put "Wilson line operators" which are 1-dimensional submanifolds labelled by a representation R of G.



When $Y = S^3$, these line operators are knots (and links) "colored by" representations, and the partition function Z_{CS} gives a knot invariant (often called quantum knot invariant).

Quantum link invariants

Quantum link invariants that will be relevant to this talk:

- colored Jones polynomials J_n: $G = SU(2)$, $R = V_n = \mathrm{Sym}^{n-1}\square$,
- colored HOMFLY-PT polynomials H_n: $G = SU(N)$, $R = \mathrm{Sym}^{n-1}\square$
- Alexander polynomial Δ: $G = U(1|1)$, $R = \square$

They are polynomials in $q = e^{\frac{2\pi i}{k}}$.

Quantum link invariants are useful in defining quantum 3-manifold invariants. If Y is the 3-manifold obtained by a Dehn surgery on a link L, then the WRT invariant of Y is a certain linear combination of the colored Jones polynomials of L.

Part 1. $\hat{Z}$ and F_K

Before we start Part 1, any questions so far?

Complex Chern-Simons theory

One can study Chern-Simons theory for complex gauge groups $G_{\mathbb{C}}$, such as $SU(2)_{\mathbb{C}} = SL(2, \mathbb{C})$. Due to non-compactness, complex Chern-Simons theory is qualitatively different from compact Chern-Simons theory.
For instance,

- infinite dimensional Hilbert spaces
- generic level

$\hat{Z}$ that we will discuss in a moment can be thought of as a non-perturbative partition function of complex Chern-Simons theory or as analytic continuation of WRT invariants.

Analytic continuation of WRT invariants

Theorem (Lawrence-Zagier '99)

Let $P = \Sigma(2,3,5) = S^3_{-1}(\mathbf{3}_1^l)$ *be the Poincare homology sphere. For every root of unity* ξ*,*

$$\tau(P;\xi) = \lim_{q\to\xi} \frac{\hat{Z}(P;q)}{2(q^{\frac{1}{2}} - q^{-\frac{1}{2}})}$$

where

$$\begin{aligned}\hat{Z}(P;q) &= q^{-\frac{3}{2}}(2 - \sum_{n\geq 0} q^n \prod_{j=0}^{n-1}(1-q^{n+j})) \\ &= q^{-\frac{3}{2}}(1 - q - q^3 - q^7 + q^8 + q^{14} + \cdots)\end{aligned}$$

There are many results by Hikami and others along this line.

$\hat{Z}$ can be seen as a natural extension of these results.

$\hat{Z}$: physical definition and categorification

Using "3d-3d correspondence", [Gukov-Putrov-Vafa '16] and [Gukov-Pei-Putrov-Vafa '17] gave a physical definition of $\hat{Z}$.

$$\begin{array}{rc} \text{space-time}: & \mathbb{R} \times \mathbb{R}^4 \times T^*Y \\ & \cup \quad\ \cup \\ N \text{ M5-branes}: & \mathbb{R} \times \mathbb{R}^2 \times Y. \end{array}$$

"Compactifying" the 6d theory on Y, we get a 3d "$\mathcal{N} = 2$" theory $T[Y, SU(N)]$. The Hilbert space is doubly graded (coming from two $U(1)$ symmetries of $\mathbb{R}^4$), and $\hat{Z}$ is the graded Euler characteristic of the Hilbert space.

$$\text{“}\hat{Z}_b(Y; q) = \sum_{i,j} (-1)^i q^j \dim \mathcal{H}^{i,j}_{BPS}(Y; b)\text{”}$$

b is a certain choice of vacua on the boundary.

They conjectured that in the limit $q \to e^{\frac{2\pi i}{k}}$, certain linear combination of $\hat{Z}_b(Y; q)$ gives the WRT invariant of Y.

Takeaways from the physical definition

For our purposes, the takeaways from the physical definition are that

- $\hat{Z}$ should be categorifiable, and that
- we need to decompose WRT invariants into a number of blocks to analytically continue.

Likorish, Wallance, and Kirby

[Gukov-Pei-Putrov-Vafa '17] also gave a mathematical definition of $\hat{Z}$ for negative definite plumbed 3-manifolds.

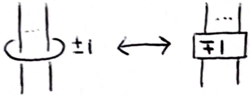
Before getting into that, let's first recall some classical results in topology:

Theorem (Likorish, Wallace '60s)

Any closed, orientable, connected 3-manifold can be obtained by performing Dehn surgery on a link in S^3.

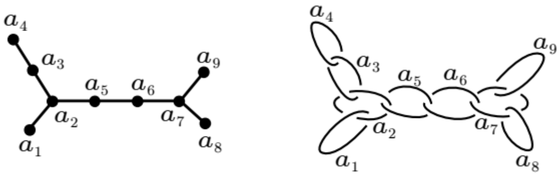
Theorem (Kirby '70s)

Two 3-manifolds obtained by Dehn surgery on links L, L' respectively are homeomorphic iff L and L' are related by a sequence of Kirby moves.

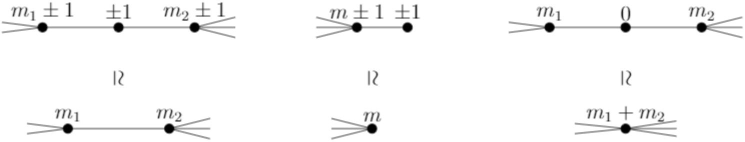


Plumbed 3-manifolds

For any tree Γ whose vertices are labelled by integers (called the plumbing graph), we can associate a 3-manifold, called the plumbed 3-manifold Y_Γ.



Neumann moves are Kirby moves for plumbing graphs.



$\hat{Z}$ for negative definite plumbed 3-manifolds

Let M be the adjacency matrix (linking matrix) of Γ. Then Y_Γ is said to be negative definite if M is.

Definition (Gukov-Pei-Putrov-Vafa '17)

Let $G = SU(2)$. For a negative definite plumbed 3-manifold Y_Γ,

$$\hat{Z}_b(Y_\Gamma) = \oint \prod_{v \in V} \frac{dx_v}{2\pi i x_v} \left(\prod_{v \in V} (x_v^{\frac{1}{2}} - x_v^{-\frac{1}{2}})^{2-\deg v} \sum_{\ell \in 2M\mathbb{Z}^V + b} q^{-\frac{1}{4}(\ell, M^{-1}\ell)} x^{\frac{\ell}{2}} \right).$$

Theorem (Gukov-Pei-Putrov-Vafa '17, Gukov-Manolescu '19)

$\hat{Z}$ is invariant under Neumann moves, and therefore is an invariant of negative definite plumbed 3-manifolds.

$\hat{Z}$ for negative definite plumbed 3-manifolds (cont.)

It's not complicated! What it really means is

1. Start from the integrand $\prod_{v\in V}(x_v^{\frac{1}{2}} - x_v^{-\frac{1}{2}})^{2-\deg v}$
2. Expand it "symmetrically", e.g.

$$(x^{\frac{1}{2}} - x^{-\frac{1}{2}})^{-1} = \frac{1}{2}\left(\cdots + x^{-\frac{3}{2}} + x^{-\frac{1}{2}} - x^{\frac{1}{2}} - x^{\frac{3}{2}} - \cdots\right)$$

3. Apply "Laplace transform"

$$\prod_{v\in V} x_v^{\ell_v} \mapsto \begin{cases} q^{-\frac{(\ell, M^{-1}\ell)}{4}} & \text{if } \ell \in 2M\mathbb{Z}^V + b \\ 0 & \text{otherwise} \end{cases},$$

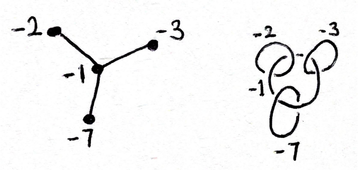
   and up to normalization we get $\hat{Z}_b(Y;q)$.

The negative-definiteness ensures convergence of the series.

$\hat{Z}$ examples

For the Poincare homology sphere $P = \Sigma(2,3,5)$, get the same series studied by Lawrence-Zagier.

For a Brieskorn homology sphere $\Sigma(2,3,7)$,



$$\hat{Z}(\Sigma(2,3,7)) = \sum_{n\geq 0} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{\prod_{j=1}^{n}(1-q^{n+j})} = 1 - q - q^5 + q^{10} - q^{11} + q^{18} + q^{30} - \cdots$$

What if M is not negative definite?

Sometimes, we can make sense of $\hat{Z}$ for plumbed 3-manifolds which are not negative definite. For instance, [Cheng-Chun-Ferrari-Gukov-Harrison '18] observed

$$\hat{Z}(-\Sigma(2,3,7)) = \sum_{n\geq 0} \frac{(-1)^n q^{-\frac{n(n+1)}{2}}}{\prod_{j=1}^{n}(1-q^{-n-j})} = \sum_{n\geq 0} \frac{q^{n^2}}{\prod_{j=1}^{n}(1-q^{n+j})}$$
$$= 1 + q + q^3 + q^4 + q^5 + 2q^7 + q^8 + 2q^9 + q^{10} + \cdots .$$

This is the seventh order mock theta function $\mathcal{F}_0(q)$ studied by Ramanujan.

The same result was obtained by [Gukov-Manolescu '19] using surgery and by [Cheng-Ferrari-Sgroi '19] using indefinite theta function.

b is a Spin^c-structure

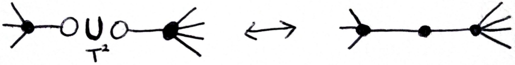
Remark (Gukov-Manolescu '19)

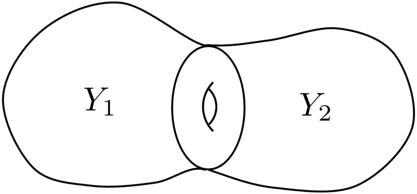
The additional label b takes values in $\mathrm{Spin}^c(Y_\Gamma)$, which is an $H_1(Y_\Gamma)$-torsor. Also, $\hat{Z}_b$ is invariant under conjugation of the Spin^c-structure b.

So, (conjecturally) $\hat{Z}$ should be a TQFT for 3-manifolds decorated with Spin^c-structures.

Cutting into pieces

If $\hat{Z}$ is really a TQFT, we should be able to compute it from a surgery description of a 3-manifold. [Gukov-Manolescu '19] studied exactly this problem, by cutting 3-manifolds into pieces with a torus boundary.





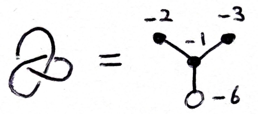
$$\hat{Z} = \sum_{n_v} \oint \frac{dx_v}{2\pi i x_v} \prod_{\text{vertices}} (\cdots) \prod_{\text{edges}} (\cdots)$$

The Hilbert space associated to a torus is infinite-dimensional, whose basis roughly correspond to $\mathbb{Z} \times \mathbb{Z}$, parametrized by n_v and the degree of x_v.

$\hat{Z}$ for knot complements: F_K

When our 3-manifold is $S^3 \setminus K$, we will write $\hat{Z}(S^3 \setminus K) =: F_K(x, q)$. The additional parameter x is a boundary condition, parametrizing the holonomy eigenvalue along the meridian in complex Chern-Simons theory.

Example: the trefoil knot



$$F^+_{\mathbf{3}_1^r}(x, q) = \sum_{m \geq 1} \left(\frac{12}{m}\right) q^{\frac{m^2+23}{24}} x^{\frac{m}{2}} = -q x^{\frac{1}{2}} + q^2 x^{\frac{5}{2}} + q^3 x^{\frac{7}{2}} - q^6 x^{\frac{11}{2}} + \cdots$$

Sometimes we write $F_K(x, q) = \frac{1}{2}\left(F^+_K(x, q) - F^+_K(x^{-1}, q)\right)$ to make the Weyl symmetry manifest.

Melvin-Morton-Rozansky expansion

[Gukov-Manolescu '19] stated a conjecture on F_K for general knot K. Before stating their conjecture, let's recall some well-known results in quantum topology.

Theorem (Bar-Natan-Garoufalidis, Rozansky '90s)

The colored Jones polynomials have the following asymptotic expansion

$$J_n(K; q = e^{\hbar}) = \sum_{j \geq 0} \frac{P_j(x)}{\Delta_K(x)^{2j+1}} \frac{\hbar^j}{j!}$$

where $P_j(x) \in \mathbb{Z}[x, x^{-1}]$, $P_0 = 1$, *and* $x = q^n = e^{n\hbar}$.

This expansion is called Melvin-Morton-Rozansky expansion.

Quantum A-polynomial

We also need to recall what quantum A-polynomial is.

Theorem (Garoufalidis-Le '03)

Colored Jones polynomials are q-holonomic. That is, there is a q-difference operator

$$\hat{A}_K(\hat{x}, \hat{y}, q) = \sum_{0 \le j \le d} a_j(\hat{x}, q)\hat{y}^j, \quad \hat{y}\hat{x} = q\hat{x}\hat{y}$$

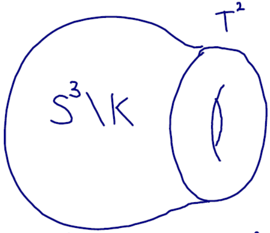
annihilating the colored Jones generating function

$$\sum_{n \ge 0} J_n(K; q) y^{-n},$$

where $\hat{y} y^{-n} = y^{-n+1}$ *and* $\hat{x} y^{-n} = q^n y^{-n}$.

Quantum A-polynomial (cont.)

Independently, the physical interpretation of the quantum A-polynomial was given by [Gukov '03].



$$\mathcal{M}_{flat}(S^3\backslash K) \subset \mathcal{M}_{flat}(T^2) = \frac{\mathbb{C}^\times \times \mathbb{C}^\times}{\mathbb{Z}_2}$$

Gukov-Manolescu conjecture

Conjecture (Gukov-Manolescu '19)

The Melvin-Morton-Rozansky expansion of colored Jones polynomials

$$(x^{\frac{1}{2}} - x^{-\frac{1}{2}})J_n(K; q = e^{\hbar}) \xlongequal{x=q^n} (x^{\frac{1}{2}} - x^{-\frac{1}{2}}) \sum_{j \geq 0} \frac{P_j(x)}{\Delta_K(x)^{2j+1}} \frac{\hbar^j}{j!}$$

can be resummed into a two-variable series $F_K(x, q)$ with integer coefficients.

Moreover, it is annihilated by the quantum A-polynomial

$$\hat{A}_K(\hat{x}, \hat{y}, q) F_K(x, q) = 0.$$

In particular, $\lim_{q \to 1} F_K(x, q) = \frac{x^{\frac{1}{2}} - x^{-\frac{1}{2}}}{\Delta_K(x)}$.

Surgery formula

Let $S^3_{p/r}(K)$ be the 3-manifold obtained by the p/r-surgery on $K \subset S^3$.

Conjecture (Gukov-Manolescu '19)

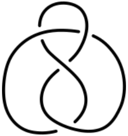
There is a surgery formula

$$\hat{Z}_b(S^3_{p/r}(K)) = \oint \frac{dx}{2\pi i x}\left((x^{\frac{1}{2r}} - x^{-\frac{1}{2r}}) F_K(x,q) \sum_{u \in \frac{p}{r}\mathbb{Z} + \frac{b}{r}} q^{-\frac{r}{p}u^2} x^u \right)$$

provided the right hand side converges.

F_K example

The figure-eight knot:



$F^{+}_{\mathbf{4}_1}(x,q) = x^{\frac{1}{2}} + 2x^{\frac{3}{2}} + (q+3+q^{-1})x^{\frac{5}{2}} + (2q^2+2q+5+2q^{-1}+2q^{-2})x^{\frac{7}{2}} + \cdots .$

Doing -1-surgery on the figure-eight knot, we get

$$\hat{Z}(S^3_{-1}(\mathbf{4}_1)) = 1 + q + q^3 + q^4 + q^5 + 2q^7 + q^8 + 2q^9 + q^{10} + \cdots ,$$

confirming the previous observation that $\hat{Z}(-\Sigma(2,3,7)) = \mathcal{F}_0(q)$, in a highly non-trivial way!

F_K as non-perturbative complex Chern-Simons partition function

In [Dimofte-Gukov-Lenells-Zagier '09], they presented a method to compute all-loop partition functions in perturbative Chern-Simons theory with gauge group $SL(2,\mathbb{C})$. For each branch $y^{(\alpha)}(x)$ of $A_K(x,y)=0$,

$$Z_{CS}^{(\alpha)}(S^3\setminus K;x,\hbar)=e^{\frac{1}{\hbar}\left(S_0^{(\alpha)}(x)+S_1^{(\alpha)}(x)\hbar+S_2^{(\alpha)}(x)\hbar^2+\cdots\right)}.$$

The F_K can be thought of as the non-perturbative $SL(2,\mathbb{C})$-Chern-Simons partition function for the abelian branch $y^{(ab)}=1$ in a sense that

$$F_K(x,q)\overset{q=e^\hbar}{=\!=}Z_{CS}^{(ab)}(S^3\setminus K;x,\hbar).$$

Knots as closure of braids

One promising approach toward a mathematical definition of F_K seems to be the use of R-matrix.

Before explaining this approach, let's recall some classical theorems.

Theorem (Alexander 1920s)

Every knot is a closure of a braid.

Theorem (Markov 1930s)

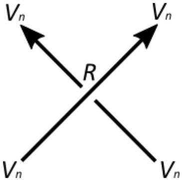
Two knots obtained by closure of the braids β, β' respectively are equivalent iff β and β' are related by a sequence of Markov moves.

Quantum groups and R-matrix

$U_q(\mathfrak{sl}_2)$ is the associative algebra over $\mathbb{C}(q^{\frac{1}{2}})$ with generators $e, f, q^{\frac{h}{2}}$ and relations

$$q^{\frac{h}{2}} e q^{-\frac{h}{2}} = qe, \quad q^{\frac{h}{2}} f q^{-\frac{h}{2}} = q^{-1} f, \quad [e, f] = \frac{q^{\frac{h}{2}} - q^{-\frac{h}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}.$$

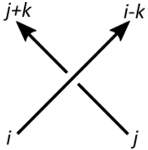
For any V_n, there is an R-matrix $R : V_n \otimes V_n \to V_n \otimes V_n$ that induces a representation of the braid group B_m on $V_n^{\otimes m}$.



The colored Jones polynomial J_n of a given braid closure is the quantum trace of the endomorphism of $V_n^{\otimes m}$ for the braid.

R-matrix at large n

Taking the limit $n \to \infty$ while keeping $x = q^n$ fixed, we get the large color R-matrix for Verma modules V_∞.

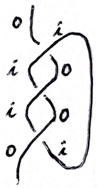


$$= x^{-\frac{1}{2}} \begin{bmatrix} i \\ k \end{bmatrix} \prod_{1 \le l \le k} \left(1 - x^{-1} q^{j+l}\right) \cdot x^{-\frac{(i-k)+j}{2}} q^{(i-k)j + \frac{(i-k)k}{2} + \frac{(i-k)+j+1}{2}}$$

Theorem (P. '20)

For positive braid knots, the conjecture of Gukov-Manolescu on the existence of F_K is true, and it can be computed using the large color R-matrix.

R-matrix at large n (cont.)



Main idea of the proof: the infinite sum converges when only positive crossings appear, and it is invariant under positive stabilization. Now the theorem follows from the following theorem:

Theorem (Etnyre-Van Horn-Morris '11)

Given a positive braid knot, any two positive braid presentations are related by a sequence of transverse Markov moves (i.e. not using negative stabilization).

There are a lot more examples

Also in [P. '20]: experimental computation of F_K for a variety of knots, including positive double twist knots and the Whitehead link.

In the classical limit $q \to 1$, F_L converges to the inverse of the Alexander-Conway function $\nabla_L(x_1, \cdots, x_m)$:

$$\lim_{q \to 1} F_L(x_1, \cdots, x_m, q) = \frac{1}{\nabla_L(x_1, \cdots, x_m)}.$$

Part 2. Large N

Before moving on to Part 2, any questions so far?

F_K for $SU(N)$

Now we turn our attention to $G = SU(N)$ (it was $G = SU(2)$ so far).

Theorem (P. '19)

The results of Gukov-Manolescu can be generalized to $SU(N)$ (or in fact any connected, simply-connected, semisimple Lie group).
In particular, there is an explicit expression for $\hat{Z}^{SU(N)}$ of negative definite plumbed 3-manifolds and $F_K^{SU(N)}(x_1, \cdots, x_{N-1}, q)$ of torus knots.

The main point in this generalization is to use the Weyl denominator $\prod_{\alpha\in\Delta^+}(x^{\frac{\alpha}{2}} - x^{-\frac{\alpha}{2}})$ instead of $x^{\frac{1}{2}} - x^{-\frac{1}{2}}$.

The classical limit can be factorized into positive roots:

$$\lim_{q\to 1} F_K^G(x_1, \cdots, x_r, q) = \prod_{\alpha\in\Delta^+} \frac{x^{\frac{\alpha}{2}} - x^{-\frac{\alpha}{2}}}{\Delta_K(x^\alpha)}.$$

Specialization to symmetric representations

Consider the specialization

$$F_K^{SU(N),sym}(x,q) := F_K^{SU(N)}(x_1,\cdots,x_{N-1},q)\Big|_{x_1=x,x_2=\cdots=x_{N-1}=q}.$$

It is experimentally checked for some torus knots that

$$\hat{A}_K(\hat{x},\hat{y},a=q^N,q)F_K^{SU(N),sym}(x,q)=0,$$

where $\hat{A}_K(\hat{x},\hat{y},a,q)$ is the q-difference equation annihilating the (symmetrically) colored HOMFLY-PT generating function.

A natural question: Is there an a-deformation of F_K?

a-deformation of F_K

Answer: Yes! [Ekholm-Gruen-Gukov-Kucharski-P.-Sułkowski '20]

Explicit expression for $(2, 2p+1)$-torus knots, and experimental calculation for the figure-eight knot

Conjecture (Ekholm-Gruen-Gukov-Kucharski-P.-Sułkowski '20)

There exists a three-variable function $F_K(x, a, q)$ that interpolates $SU(N)$-F_K's in the sense that $F_K(x, q^N, q) = F_K^{SU(N),sym}(x, q)$. Moreover, the series $F_K(x, a, q)$ has the following properties :

1. $\hat{A}_K(\hat{x}, \hat{y}, a, q) F_K(x, a, q) = 0$
2. $F_K(x^{-1}, a, q) = F_K(a^{-1}q^2 x, a, q)$ *(Weyl symmetry)*

3a. $F_K(x, q^N, q)\Big|_{q \to 1} = \Delta_K(x)^{1-N}$

3b. $F_K(x, q, q) = 1$

3c. $F_K(x, 1, q) = \Delta_K(q^{-1}x)$

Topological strings

Recall the well-known result by [Ooguri-Vafa '00]: colored HOMFLY-PT generating function is the open string partition function of the following system, where L_K is the conormal Lagrangian.

$$\begin{aligned} \text{space-time}: \quad & \mathbb{R} \times \mathbb{R}^4 \times T^*S^3 \\ & \qquad\ \ \cup \qquad \cup \\ N \text{ M5-branes}: \quad & \mathbb{R} \times \mathbb{R}^2 \times S^3 \\ r \text{ M5-branes}: \quad & \mathbb{R} \times \mathbb{R}^2 \times L_K. \end{aligned}$$

In the large N limit, the system becomes

$$\begin{aligned} \text{space-time}: \quad & \mathbb{R} \times \mathbb{R}^4 \times X \\ & \qquad\ \ \cup \qquad \cup \\ r \text{ M5-branes}: \quad & \mathbb{R} \times \mathbb{R}^2 \times L_K, \end{aligned}$$

where X is the resolved conifold, the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$.

a-deformed F_K and topological strings

In our situation, we choose a different Lagrangian filling, namely the knot complement Lagrangian M_K.

$$\begin{aligned} \text{space-time}: \quad & \mathbb{R} \times \mathbb{R}^4 \times T^*S^3 \\ & \qquad\ \ \cup \qquad \cup \\ N-1 \text{ M5-branes}: \quad & \mathbb{R} \times \mathbb{R}^2 \times S^3 \\ 1 \text{ M5-brane}: \quad & \mathbb{R} \times \mathbb{R}^2 \times M_K. \end{aligned}$$

A novel feature is that M_K cannot be shifted off of S^3 completely when K is not fibered, so in the large N limit, there can still be some finite number of cotangent fibers where Reeb chords can end.

Physicallly, $F_K(x, a, q)$ is the open string partition function of this system.

$$F_K(x, a, q = e^{g_s}) = e^{\frac{1}{g_s} U_K(x,a) + U_K^0(x,a) + g_s U_K^1(x,a) + g_s^2 U_K^2(x,a) + \cdots}$$

where U_K counts disks and U_K^k counts curves of Euler characteristic $\chi = -k$.

a-deformed F_K and topological strings (cont.)

One consequence of our conjecture is that

$$\langle \hat{b} \rangle|_{(y,a)=(1,1)} = \lim_{q \to 1} \frac{F_K(x, qa, q)}{F_K(x, a, q)}\bigg|_{(y,a)=(1,1)} = \Delta_K(x)^{-1}.$$

Moreover,

$$\begin{aligned}
\langle \hat{b} \rangle|_{(y,a)=(1,1)} &= \exp\left(\frac{\partial U_K(x,a)}{\partial \log a}\bigg|_{(y,a)=(1,1)} \right) \\
&= \exp\left(\int \frac{\partial \log y(x,a)}{\partial \log a}\bigg|_{(y,a)=(1,1)} d\log x \right) \\
&= \exp\left(\int -\frac{\partial_{\log a} A_K}{\partial_{\log y} A_K}\bigg|_{(y,a)=(1,1)} d\log x \right).
\end{aligned}$$

a-deformed F_K and topological strings (cont.)

This was confirmed by the following recent result!

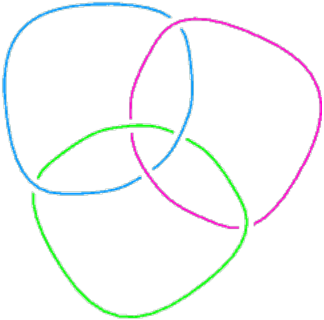
Theorem (Diogo-Ekholm '20)

$$\Delta_K(x) = (1-x)\exp\left(\int \frac{\partial_{\log a}\mathrm{Aug}_K}{\partial_{\log y}\mathrm{Aug}_K}\bigg|_{(y,a)=(1,1)} d\log x\right)$$

Their proof uses several cobordisms between moduli spaces of holomorphic curves in the setup of knot complement Lagrangian.

Lessons

- $\hat{Z}$ is an interesting object in quantum topology that is related to categorification, modularity, etc.
- It can be understood in the context of open topological strings, and it is an interesting problem to understand precisely how to count the holomorphic curves in this setting, when K is non-fibered.
- Many open questions:
  - Find a mathematical definition of $\hat{Z}$ for all 3-manifolds.
  - Categorify it!
  - Large N of $\hat{Z}$? (related to large N transition for 3-manifolds)



Thank you for your attention!