Inverted state sums for knot complements

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October 11, 2021 @ *USC Algebra Seminar*

Overview

A couple of years ago, S. Gukov and C. Manolescu conjectured that the Melvin-Morton-Rozansky expansion of the colored Jones polynomials can be resummed into a two-variable series $F_K(x, q)$, as part of a bigger program to construct a 3d TQFT $\hat{Z}$ that had been predicted from physics.

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This talk is based on [arXiv:2106.03942].

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1. Some background

Colored Jones polynomials

Let L be a link and $n \geq 1$ an integer.

The n-colored Jones polynomial $J_{L,n}(q) \in \mathbb{Z}[q, q^{-1}]$ is an invariant of the link L that can be defined in many different ways (e.g. Reshetikhin-Turaev construction, Jones-Wenzl projectors, cabling).

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$U_q(\mathfrak{sl}_2)$

Quantum $\mathfrak{sl}_2$, $U_q(\mathfrak{sl}_2)$, is the associative algebra over $\mathbb{C}(q^{\frac{1}{2}})$ generated by $E, F, K(= q^{\frac{H}{2}}), K^{-1}$ with relations

$$KE = qEK, \quad KF = q^{-1}FK, \quad [E,F] = \frac{K - K^{-1}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}.$$

For each $n \geq 1$, let V_n be the n-dimensional $U_q(\mathfrak{sl}_2)$-module with basis $\{v^0, \cdots, v^{n-1}\}$ on which the generators act by

$$Ev^j = [j]v^{j-1}, \quad Fv^j = [n-1-j]v^{j+1}, \quad Kv^j = q^{\frac{n-1-2j}{2}} v^j,$$

where $[m] := \frac{q^{\frac{m}{2}} - q^{-\frac{m}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$ for any $m \in \mathbb{Z}$.

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$U_q(\mathfrak{sl}_2)$ admits a universal R-matrix

$$R = q^{\frac{H\otimes H}{4}} \sum_{k\geq 0} q^{\frac{k(k-1)}{4}} \frac{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^k}{[k]!} (E^k \otimes F^k) \in U_q(\mathfrak{sl}_2)\hat{\otimes} U_q(\mathfrak{sl}_2).$$

Applied to V_n, we obtain an automorphism $\check{R} \in \mathrm{Aut}(V_n \otimes V_n)$ given by

$$\check{R}(v^i \otimes v^j) = \sum_{k\geq 0} q^{\frac{n^2-1}{4} - \frac{(i-k+j+1)(n-1)}{2} + (i-k)j} \times \begin{bmatrix} i \\ k \end{bmatrix}_q \prod_{1\leq l\leq k} (1-q^{-n+j+l}) v^{j+k} \otimes v^{i-k}$$

that satisfies the Yang-Baxter equation

$$\check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23}.$$

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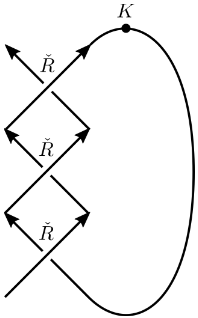
Colored Jones polynomials

Let L be a link which can be presented as the closure of a braid β. Then its colored Jones polynomial $J_{L,n}(q)$ is the quantum trace of the automorphism given by the product of R-matrices.



Melvin-Morton-Rozansky expansion

Theorem (Conjectured by Melvin-Morton and Rozansky, proved by Bar-Natan-Garoufalidis and Rozansky)

Set $q = e^{\hbar}$. Consider the limit where $\hbar \to 0$ and $n \to \infty$ while $n\hbar$ is fixed. In this large-color limit, the colored Jones polynomial has the following expansion:

$$J_{K,n}(q) = \frac{1}{\Delta_K(x)} + \frac{P_1(x)}{\Delta_K(x)^3}\hbar + \frac{P_2(x)}{\Delta_K(x)^5}\frac{\hbar^2}{2!} + \cdots ,$$

where $x = e^{n\hbar}$, $\Delta_K(x)$ is the Alexander polynomial, and $P_j(x) \in \mathbb{Z}[x, x^{-1}]$ are Laurent polynomials invariant under Weyl symmetry $x \leftrightarrow x^{-1}$.

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2. Gukov-Manolescu conjecture and $\hat{Z}$

Gukov-Manolescu conjecture

Conjecture (Gukov-Manolescu 2019)

The MMR expansion of the colored Jones polynomials can be resummed into a two-variable series

$$(x^{\frac{1}{2}} - x^{-\frac{1}{2}}) \sum_{j \geq 0} \frac{P_j(x)}{\Delta_K(x)^{2j+1}} \frac{\hbar^j}{j!} = F_K(x, q)$$

where the left-hand side is understood as the average of its power series expansions near $x = 0$ and $x = \infty$, and $F_K(x, q)$ is a formal series of the form $\frac{1}{2} \sum_{m \in 2\mathbb{Z}+1} f_m(q) x^{\frac{m}{2}}$ with $f_m(q) \in \mathbb{Z}((q))$.

Moreover, $\hat{A}_K(\hat{x}, \hat{y}, q) F_K(x, q) = 0$, where $\hat{A}_K$ is the quantum A-polynomial for the unreduced colored Jones polynomials of K.

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Gukov-Manolescu conjecture

$$
\begin{aligned}
&2F_K(x, e^\hbar) \; = \\
&= \left(x^{1/2} - \frac{1}{x^{1/2}} + 2x^{3/2} - \frac{2}{x^{3/2}} + 5x^{5/2} - \frac{5}{x^{5/2}} + 13x^{7/2} - \frac{13}{x^{7/2}} + 34x^{9/2} - \frac{34}{x^{9/2}} + 89x^{11/2} - \frac{89}{x^{11/2}} + 233x^{13/2} - \frac{233}{x^{13/2}} + \ldots\right) \\
&\qquad + \hbar^2 \left(x^{5/2} - \frac{1}{x^{5/2}} + 10x^{7/2} - \frac{10}{x^{7/2}} + 64x^{9/2} - \frac{64}{x^{9/2}} + 331x^{11/2} - \frac{331}{x^{11/2}} + 1505x^{13/2} - \frac{1505}{x^{13/2}} + \ldots\right) \\
&+ \hbar^4 \left(\frac{x^{5/2}}{12} - \frac{1}{12x^{5/2}} + \frac{17x^{7/2}}{6} - \frac{17}{6x^{7/2}} + \frac{142x^{9/2}}{3} - \frac{142}{3x^{9/2}} + \frac{6115x^{11/2}}{12} - \frac{6115}{12x^{11/2}} + \frac{50057x^{13/2}}{12} - \frac{50057}{12x^{13/2}} + \ldots\right) \\
&+ \hbar^6 \left(\frac{x^{5/2}}{360} - \frac{1}{360x^{5/2}} + \frac{13x^{7/2}}{36} - \frac{13}{36x^{7/2}} + \frac{818x^{9/2}}{45} - \frac{818}{45x^{9/2}} + \frac{154891x^{11/2}}{360} - \frac{154891}{360x^{11/2}} + \frac{472573x^{13/2}}{72} - \frac{472573}{72x^{13/2}} + \ldots\right) \\
&+ \hbar^8 \Big(\frac{x^{5/2}}{20160} - \frac{1}{20160x^{5/2}} + \frac{257x^{7/2}}{10080} - \frac{257}{10080x^{7/2}} + \frac{10781x^{9/2}}{2520} - \frac{10781}{2520x^{9/2}} + \frac{916439x^{11/2}}{4032} - \frac{916439}{4032x^{11/2}} + \frac{19085471x^{13/2}}{2880} \\
&\qquad\qquad - \frac{19085471}{2880x^{13/2}} + \ldots\Big) \\
&+ \hbar^{10} \Big(\frac{x^{5/2}}{1814400} - \frac{1}{1814400x^{5/2}} + \frac{41x^{7/2}}{36288} - \frac{41}{36288x^{7/2}} + \frac{9608x^{9/2}}{14175} - \frac{9608}{14175x^{9/2}} + \frac{147178651x^{11/2}}{1814400} - \frac{147178651}{1814400x^{11/2}} \\
&\qquad\qquad + \frac{47916623x^{13/2}}{10368} - \frac{47916623}{10368x^{13/2}} + \ldots\Big) \\
&\qquad\qquad\qquad + \ldots
\end{aligned}
$$

Figure: Figure-eight knot example from [Gukov-Manolescu]

Gukov-Manolescu conjecture

$$
\begin{aligned}
f_1 &= 1,\\
f_3 &= 2,\\
f_5 &= 1/q + 3 + q,\\
f_7 &= 2/q^2 + 2/q + 5 + 2q + 2q^2,\\
f_9 &= 1/q^4 + 3/q^3 + 4/q^2 + 5/q + 8 + 5q + 4q^2 + 3q^3 + q^4,\\
f_{11} &= 2/q^6 + 2/q^5 + 6/q^4 + 7/q^3 + 10/q^2 + 10/q + 15 + 10q + 10q^2 + 7q^3 + 6q^4 + 2q^5 + 2q^6,\\
f_{13} &= 1/q^9 + 3/q^8 + 4/q^7 + 7/q^6 + 11/q^5 + 15/q^4 + 18/q^3 + 21/q^2 + 23/q + 27 + 23q\\
&\quad + 21q^2 + 18q^3 + 15q^4 + 11q^5 + 7q^6 + 4q^7 + 3q^8 + q^9,
\end{aligned}
$$

Figure: Figure-eight knot example from [Gukov-Manolescu], continued

$\hat{Z}$: a TQFT for complex Chern-Simons theory

In fact, Gukov-Manolescu conjecture is part of a bigger program to construct a 3d TQFT $\hat{Z}$ predicted from physics [Gukov-Putrov-Vafa, Gukov-Pei-Putrov-Vafa, Gukov-Marino-Putrov].

Simply put, $\hat{Z}$ assigns a q-series to a 3-manifold decorated with a spin^c-structure. Physics provides a variety of ways to think of $\hat{Z}$:

- One way is as resurgence of perturbative series associated to abelian flat connections in the complex Chern-Simons theory.
- Another way is as the BPS index of a 3d $\mathcal{N} = 2$ theory $T[Y]$ on $\mathbb{C} \times_q S^1$. The space of BPS states in this theory provides a natural categorification of $\hat{Z}$

$$\hat{Z}_{Y,\mathfrak{s}}(q) = \sum_{i,j} (-1)^i q^j \mathcal{H}^{i,j}_{\mathrm{BPS}}(Y, \mathfrak{s}).$$

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	numerical invariants	homological invariants
knots	colored Jones polynomials	KhR homology
3-manifolds	$\hat{Z}$	$\mathcal{H}_{\mathrm{BPS}}$

Table: Analogue of Khovanov homology for 3-manifolds

$\hat{Z}$: a TQFT for complex Chern-Simons theory

The two-variable series $F_K(x, q)$ is simply $\hat{Z}$ for the knot complement $S^3 \setminus K$.

There are some surgery formulas allowing us to compute $\hat{Z}$ for 3-manifolds obtained as some Dehn surgery on a link. An explicit example:

$$\hat{Z}_{-\Sigma(2,3,7)} = \hat{Z}_{S^3_{-1}(4_1)} = \left[(x^{\frac{1}{2}} - x^{-\frac{1}{2}})F_{4_1}(x, q)\right]_{x^u \mapsto q^{u^2}}$$
$$= \sum_{n \geq 0} \frac{q^{n^2}}{\prod_{1 \leq k \leq n}(1 - q^{n+k})}$$

This is Ramanujan's mock theta function $F_0(q)$ of order 7.

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3. Inverted state sums

Large-color R-matrix

Since the MMR expansion is about the large-color asymptotics of the colored Jones polynomials, it is natural to study the corresponding large-color limit of the R-matrix.

In the large-color limit, the representation V_n becomes a Verma module V_∞ with generic highest (or lowest) weight.

The highest weight Verma module with highest weight $\lambda = \log_q x - 1$ has a basis $\{v^j\}_{j\geq 0}$ on which the generators of $U_q(\mathfrak{sl}_2)$ act by

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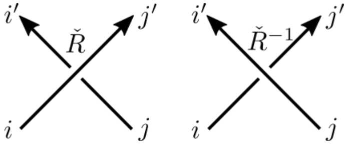
Large-color R-matrix

Call the R-matrix for these Verma modules the large-color R-matrix.

$$\check{R}(x_1,x_2)_{i,j}^{i',j'} = \delta_{i+j,i'+j'} q^{(j+\frac{1}{2})(j'+\frac{1}{2})} x_1^{\frac{-i'-j-1}{4}} x_2^{\frac{i-3j'-1}{4}} \times \begin{bmatrix} i \\ j' \end{bmatrix}_q \prod_{1\leq l\leq i-j'} (1-q^{j+l}x_2^{-1})$$



Extending the R-matrix

While the indices $i, j, i', j' \in \mathbb{Z}_{\geq 0}$ represent the basis vectors $\{v^i\}_{i \geq 0}$ of the highest weight Verma module V_∞, the R-matrix element $\check{R}(x_1, x_2)_{i,j}^{i',j'}$ makes sense for all $i, j, i', j' \in \mathbb{Z}$.

$$\check{R}(x,y)_{i,j}^{i',j'} = \begin{cases} \delta_{i+j,i'+j'} q^{(j+\frac{1}{2})(j'+\frac{1}{2})} x^{-\frac{i'+j+1}{4}} y^{-\frac{3j'-i+1}{4}} \begin{bmatrix} i \\ i-j' \end{bmatrix}_q \prod_{1 \leq l \leq i-j'} (1-q^{j+l} y^{-1}) & \text{if } i \geq j' \geq 0 \text{ or } 0 > i \geq j' \\ \delta_{i+j,i'+j'} q^{(j+\frac{1}{2})(j'+\frac{1}{2})} x^{-\frac{i'+j+1}{4}} y^{-\frac{3j'-i+1}{4}} \begin{bmatrix} i \\ j' \end{bmatrix}_q \frac{1}{\prod_{0 \leq l \leq j'-i-1} (1-q^{j-l} y^{-1})} & \text{if } j' \geq 0 > i \\ 0 & \text{otherwise} \end{cases}$$

Extending the R-matrix

While the indices $i, j, i', j' \in \mathbb{Z}_{\geq 0}$ represent the basis vectors $\{v^i\}_{i\geq 0}$ of the highest weight Verma module V_∞, the R-matrix element $\check{R}(x_1, x_2)_{i,j}^{i',j'}$ makes sense for all $i, j, i', j' \in \mathbb{Z}$.

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Meaning of the extension

Consider the Verma module with height weight $\lambda = \log_q x - 1$

$$\cdots \underset{F}{\overset{E}{\rightleftharpoons}} V_\infty(\lambda - 2) \underset{F}{\overset{E}{\rightleftharpoons}} V_\infty(\lambda).$$

This can be naturally extended to the principal series module

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The upper-half part of this module,

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Therefore, changing the domain of an index from $\mathbb{Z}_{\geq 0}$ to $\mathbb{Z}_{<0}$ corresponds to passing from the highest weight Verma module to the lowest weight Verma module.

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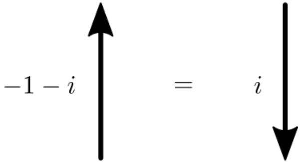

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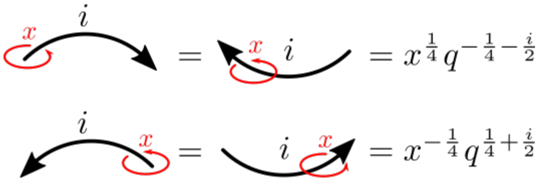
The pairing and copairing are given by



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The large-color R-matrix enjoys various symmetries that are natural from the diagrammatic point of view. For instance,

$$\check{R}(x_1, x_2)^{i', -1-j'}_{-1-i, j} = q^{\frac{i-j'}{2}} \check{R}^{-1}(x_2, x_1^{-1})^{i,i'}_{j,j'},$$

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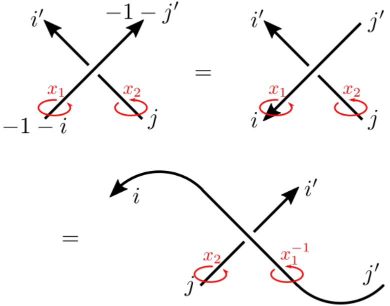

Inverted state sum

Let L be an oriented link, and let D be its diagram as a $(1,1)$-tangle.

An inversion datum on D is a map that assigns a sign ($+$ or $-$) to each segment of D in such a way that on each crossing the number of signs of each type are preserved.

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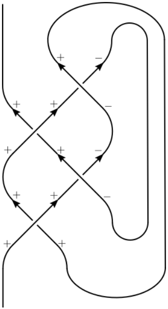

Inverted state sum

A diagram with an inversion datum determines a model of state sum that we call an inverted state sum.

That is, we replace each crossing by the corresponding R-matrix element and each local maximum and minimum in vertical direction by the corresponding pairing and copairing, respectively. Then we sum over all the internal indices; whenever the segment has $-$ sign, we flip range of summation to $\mathbb{Z}_{<0}$ (or equivalently, change the orientation of the segment).

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Let's say a link L is "nice" if it admits a link diagram with an inversion datum such that the inverted state sum is absolutely convergent in $\mathbb{Z}[q, q^{-1}][[x^{-1}]]$.

Theorem (P. 2021)

Gukov-Manolescu conjecture is true for any nice link.

- Homogeneous braid links are nice.
- All fibered knots up to 10 crossings are nice.

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Digression: connection to enumerative geometry

In fact, this conjecture is expected from a completely different point of view, namely enumerative geometry.

From enumerative geometry point of view, $F_K(x, a, q)$, a 1-parameter deformation of $F_K(x, q)$, is the count of open holomorphic curves in the resolved conifold with the knot complement Lagrangian. Knots-quivers correspondence for knot complements predicts that all such holomorphic curves arise from a finite collection of basic holomorphic disks.

When the knot is fibered, the knot complement Lagrangian can be completely shifted off from the zero section of T^*S^3, and a positive area argument suggests that all the basic disk should have positive x-degree, implying that the coefficients of $F_K(x, a, q)$ as a power series in x are rational functions in a and q.

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Sketch of proof of the theorem

The main idea is to use Foata-Zeilberger formula.

A cycle on an oriented graph is called primitive if it is not a power of any other cycle. A multi-cycle is an unordered tuple of cycles. A multi-cycle is called primitive if each of the cycles are primitive.

Theorem (Foata-Zeilberger)

Let G be an oriented graph with weighted edges. Then the weighted sum of all primitive multi-cycles on G is

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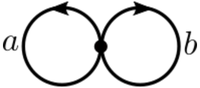
Example:



$$
\begin{aligned}
&1\\
&+(a)+(b)\\
&+(ab)+(a)(b)+(a)^2+(b)^2\\
&+(a^2b)+(ab^2)+(a)(ab)+(b)(ab)+(a)^3+(a)^2(b)+(a)(b)^2+(b)^3\\
&+\cdots\\
&=\frac{1}{1-(a+b)}.
\end{aligned}
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In the classical limit $q = 1$, the inverted state sum becomes the weighted sum of all primitive multi-cycles on the oriented graph whose vertices are the set of internal segments. Therefore,

$$Z^{\text{inv}}(D) = \frac{1}{\det(I - \mathcal{B}_{\text{inv}})}.$$

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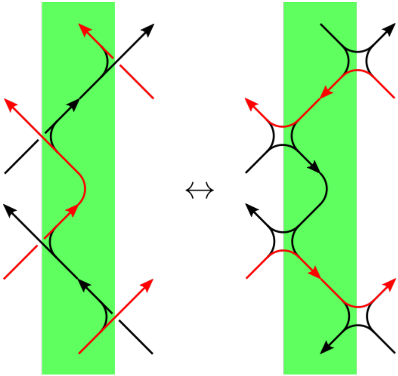

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That is, we use parametrized R-matrix. Higher perturbative terms of the R-matrix can be obtained by applying a differential operator to the parametrized R-matrix and then specializing the parameters.

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